ON THE STABILITY OF STEADY MOTIONS OF A HEAVY BODY OF REVOLUTION ON AN ABSOLUTELY ROUGH HORIZONTAL PLANE

(OB USTOICHIVOSTI STATSIONARNYKH DVIZHENII TIAZHELOGO TELA VRASHCHENIIA NA ABSOLIUTNO SHEROKHOVATOI GORIZONTAL'NOI PLOSKOSTI)

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The stability of a spinning top with spherical support has been studied in [1] by means of a Liapunov function constructed from the integral of energy and the integrals of Zhelle and Chaplygin [2]. The stability of a rectilinearly rolling disk with a gyroscope is investigated in [3], and the stability of arbitrary steady motions of the disk on a plane is considered in [4]. For these studies the hypergeometric solutions of Appel and Korteveg [2] were used. The stability of the steady motions (in which the axis of the body can be arranged vertically and horizontally) of a body with a gyroscope, constrained by an arbitrary surface of rotation, is investigated in [4].

Thereby integrals depending linearly on the angular velocities are indicated and used, and the force function is assumed to be analytical, which guarantees the analytic feature of the solution. For the construction of the Liapunov function in the neighborhood of the steady motion the first two terms of the series are computed.

In the present paper we obtain the necessary and sufficient condition of stability of all steady motions of a heavy homogeneous body constrained by an arbitrary surface of rotation by using, for a Liapunov function, the sum of the squares of integrals [5]. The investigation of the sign-definiteness conditions for this function did not require an explicit computation of the linear integrals.

We shall consider a heavy rigid body, rolling without slipping on a horizontal plane and constrained by the surface of rotation which has the axis ζ. Let the body be dynamically symmetrical with respect to the axis and let it bear the rotor of a gyroscope mounted so that it can rotate freely on the axis ζ. We shall introduce two systems of coordinates, $OXYZ$ which is fixed, and $G\xi\eta\zeta$ which is mobile with its origin at G, center of gravity of the system. The axis $G\xi$ is directed in the plane of the vertical meridian perpendicularly to the axis $G\zeta$, and the axis $G\eta$ perpendicularly to the plane of the vertical meridian. We shall denote by σ the angle ζNM of the axis of the body with the horizontal tangent NM of its meridian GF; by p, q, r, respectively the components of the angular velocity of the body on the axes ξ, η, ζ. The principal moment of momentum of the gyroscope around its axis is denoted by s; according to the conditions of the problem s = const.

Let M be the mass of the system; A is its moment of inertia about the axes $G\xi$, $G\eta$; B is the moment of inertia of the body alone, and B^1 that of the gyroscope alone about the axis of symmetry. Let also $(0, \xi(\alpha), \eta(\alpha))$ be the coordinates of the point of contact of the body and the plane.

As it was shown by Chaplygin, three equations of motion can be obtained for $\alpha \neq \frac{1}{2}\pi$ in the form [2]

$$\frac{d}{dt}(p\zeta - r\xi) + pq(\zeta \tan\alpha - \zeta) = \frac{B}{\xi M}\frac{dr}{dt} \qquad \left(q = -\frac{d\alpha}{dt}\right) \qquad (1)$$

$$A\frac{dp}{dt} + (Br + s + Ap\tan\alpha)q = -\frac{B\zeta}{\xi}\frac{dr}{dt}$$

Furthermore, the mechnical system under consideration has an integral of energy

$$2H = Ap^2 + Br^2 + M(p\zeta - r\xi) + [A + M(\zeta^2 + \xi^2)]q^2 + 2Mgz(\alpha) = \text{const} \qquad (2)$$

Here $z(\alpha)$ is the height of the center of gravity of the body above the horizontal plane

$$\xi = z\cos\alpha - z'\sin\alpha, \qquad \zeta = -z\sin\alpha - z'\cos\alpha$$

Calculating the time derivative of (2) and taking (1) into consideration, we get the fourth equation of motion

$$[A + M(\xi^2 + \eta^2)]\frac{dq}{dt} = p(Br + s + Ap\tan\alpha) + M(\xi\xi' + \zeta\zeta')q^2 +$$

$$+ Mgz' + Mp(\zeta\tan\alpha - \xi)(p\zeta - r\xi) \qquad (3)$$

Here ξ', η', z' are the derivatives with respect to α.

For $q = -d\alpha/dt \neq 0$, the linear equations follow

$$\frac{d}{d\alpha}(p\zeta - r\xi) - p(\zeta\tan\alpha - \xi) = \frac{B}{\xi M}\frac{dr}{d\alpha}, \quad A\frac{dp}{d\alpha} - (Br + s + Ap\tan\alpha) = -\frac{B\zeta}{\xi}\frac{dr}{d\alpha}$$

Solving these with respect to the derivatives we get

$$\frac{dp}{d\alpha} = (\tan\alpha + a_1)p + a_2 r + h_1, \qquad \frac{dr}{d\alpha} = b_1 p + b_2 r + h_2 \qquad (4)$$

where

$$a_1 = \frac{B\zeta(\zeta' + \xi)}{\Delta}, \quad a_2 = \frac{B(B/M + \xi^2 + \zeta\xi')}{\Delta}, \quad b_1 = \frac{A\xi(\xi + \zeta')}{\Delta}, \quad b_2 = \xi\frac{B\xi - A\zeta'}{\Delta} \qquad (5)$$

$$h_1 = \frac{B + M\xi^2}{\Delta}, \qquad h_2 = \frac{\xi\zeta}{\Delta}, \qquad \Delta = \frac{AB}{M} + A\xi^2 + B\zeta^2 > 0$$

Let

$$p = c_1\varphi_1(\alpha) + c_2\varphi_2(\alpha) + \varphi_3(\alpha), \qquad r = c_1\psi_1(\alpha) + c_2\psi_2(\alpha) + \psi_3(\alpha) \qquad (6)$$

be the general solution of Equations (4).

Solving (6) with respect to the constants, we get two integrals [4]

$$\lambda_1(\alpha)(p - \varphi_3) + \lambda_2(\alpha)(r - \psi_3) = c_1, \qquad \mu_1(\alpha)(p - \varphi_3) + \mu_2(\alpha)(r - \psi_3) = c_2 \qquad (7)$$

Obviously, any integral $F(\alpha, p, r)$ of Equations (4) is an integral of the system (1) because if $dF/d\alpha = 0$, on the basis of (4), there follows

$$\frac{dF}{dt} = \frac{dF}{d\alpha}\frac{d\alpha}{dt} = 0$$

on the basis of (1). All the following calculations are valid if the potential energy $Mgz(\alpha)$ possesses two continuous derivatives, which, in agreement with the relations

$$\xi = z\cos\alpha - z'\sin\alpha, \qquad \zeta = -z\sin\alpha - z'\cos\alpha$$

guarantees the boundedness of ξ' and ζ', and consequently the boundedness of the coefficients of the system (4) on the interval $0 \leqslant \alpha \leqslant {}^1/_2\pi - \varepsilon^2$ (where ε is an arbitrary small quantity), and furthermore it yields an existence condition for the function $\delta H'$ in the form given below. Thus is removed the requirement of the analytic feature of $z(\alpha)$ given in [4].

Equations (1.1) and (1.3) have a partial solution

$$\alpha = \alpha_0 \neq {}^1/_2\pi, \quad q = 0, \quad p = p_0, \quad r = r_0 \qquad (8)$$

if the constants p_0, r_c, α_0 satisfy Equation

$$p_0^2 A_1 \tan\alpha_0 + (B_1 r_0 + s)\, p_0 + Mgz'(\alpha_0) = 0 \tag{9}$$

Here

$$A_1 = A - Mz\,\frac{\zeta(\alpha_0)}{\sin\alpha_0}, \qquad B_1 = A + Mz\,\frac{\xi(\alpha_0)}{\cos\alpha_0} \tag{10}$$

The positiveness of the discriminant of Equation (9), quadratic with respect to p_0 is given by a condition of existence of the solutions (8)

$$D(\alpha_0, r_0) = (B_1 r_0 + s)^2 - 4MgA_1 z'(\alpha_0)\tan\alpha_0 > 0 \tag{11}$$

Let us consider the stability of the steady motion (8). The stability of steady vertical rotations ($\alpha = \frac{1}{2}\pi$) has been investigated in [3 and 4]; the stability of all rotations has been studied for the case of the disk in [6].

Let us assume that for the undisturbed motion (8) the integrals H, c_1, c_2 take the values H°, $c_1{}^\circ$, $c_2{}^\circ$. Let us denote by x_1, x_2 the variations of the variables p, r; by q, $\delta\alpha$, the variations of q, α and by δH, δc_1, δc_2 the variations of the functions H, c_1, c_2. Now, let us consider the sum of the squares of the integrals of the equations of the disturbed motion [5]

$$V = [\delta H(x_1, x_2, q, \delta\alpha)]^2 + [\delta c_1(x_1, x_2, q, \delta\alpha)]^2 + (\delta c_2)^2$$

This positive function will be positive definite, if $\delta H > 0$ for those values of the arguments when $\delta c_1 = \delta c_2 = 0$.

In other words, Equations $\delta c_1 = \delta c_2 = 0$ yield $x_1(q, \delta\alpha)$, $x_2(q, \delta\alpha)$; substituting these values into the function δH, we get

$$\delta H(x_1(q, \delta\alpha), x_2(q, \delta\alpha), q, \delta\alpha) = \delta H^1(q, \delta\alpha)$$

If $\delta H^1(q, \delta\alpha)$ is a sign-definite function of its arguments then, and only then, the function V is positive definite. In other words, the variation of H calculated for the constants $c_1{}^\circ$, $c_2{}^\circ$, must be sign-definite. In a practical calculation the knowledge of an explicit form of the functions $c_1(x_1, \ldots, \delta\alpha)$, $c_2(x_1, \ldots, \delta\alpha)$ is not indispensible, because the quantities $dp/d\alpha$, $dr/d\alpha$ entering the variation δH are equal to the right-hand sides of Equations (4), taken on the steady solution (8). Denoting the variation of $\delta H(c_1{}^\circ, c_2{}^\circ, q, \delta\alpha) = \delta H^1$, we obtain

$$\delta H^1 = \left[\frac{\partial H^1(c_1{}^\circ, c_2{}^\circ, q, \delta\alpha)}{\partial\alpha}\right]^\circ \delta\alpha + \frac{1}{2}\left[\frac{\partial^2 H^1}{\partial q^2}\right]^\circ q^2 + \frac{1}{2}\left[\frac{\partial^2 H^1}{\partial\alpha^2}\right]^\circ (\delta\alpha)^2 + Z =$$

$$= f_1\delta\alpha + f_2 q^2 + f_3(\delta\alpha)^2 + Z$$

where Z are the terms of higher order and

$$\left[\frac{\partial^2 H^1}{\partial q\,\partial\alpha}\right]^\circ \equiv 0$$

By virtue of (4) and (8) we get

$$\frac{1}{2}\left[\frac{\partial H^1}{\partial\alpha}\right]^\circ = \left[Ap\frac{dp}{d\alpha} + 2Br\frac{dr}{d\alpha} + M(\xi\xi' + \zeta\zeta')q^2 + Mgz' + M(p\zeta - r\xi)\left(p\zeta' - r\xi' + \right.\right.$$

$$\left.\left. + \zeta\frac{dp}{d\alpha} - \xi\frac{dp}{d\alpha}\right)\right]^\circ = p_0^2 A_1 \tan\alpha + (B_1 r_0 + s)\, p_0 + Mgz'(\alpha_0) = f_1(\alpha_0)$$

By virtue of (9) we conclude that $f_1(\alpha_0) = 0$. As far as we have always

$$\left[\frac{\partial^2 H^1}{\partial q^2}\right]^\circ > 0$$

then in order to insure the positive-definiteness of δH^1 it is sufficient to have $f_3 > 0$, whereby that quantity can be computed by taking the derivative with respect to α_0 of the left-hand side of Equation (9), considering p_0, q_0 as functions of α_0 (i.e. on the basis of (4)). Making the indicated computations, we get the stability condition in the form

$$f_3 = (2A_1\tan\alpha_0 p_0 + B_1 r_0 + s)\,[(\tan\alpha_0 + a_1)\,p_0 + a_2 r_0 + h_1] + A_1 p_0^2 \sec^2\alpha_0 +$$

$$+ p_0^2 A_1' \tan\alpha_0 + B_1 p_0(b_1 p_0 + b_2 r_0 + h_2) + B_1' r_0 p_0 + Mgz(\alpha_0) > 0 \tag{12}$$

in which the parameters of motion are related by Equation (9); A_1', A_2' are the derivatives with respect to α_0 of A_1 and B_1. Since

$$dH^1(c_1^\circ, \ldots, \alpha) / dt \equiv 0$$

must be fulfilled by virtue of (3), there follows that (3) is equivalent to Equation

$$\frac{d}{dt} H = \frac{d}{dt} [\gamma(\alpha) q^2 + \beta(\alpha)] = q [2\gamma q^{\cdot} + \gamma q^2 + \beta'] = 0 \quad \text{for} \quad q \neq 0 \ (\beta'^{\circ} = f_1 = 0)$$

It is thereby clear, that the linear approximation equation for $\delta\alpha$ is

$$2\gamma(\alpha_0)(\delta\alpha)^{\cdot\cdot} + \left[\frac{\partial^2 H^1}{\partial \alpha^2}\right]^\circ \delta\alpha = 0$$

and for $f_3 < 0$ the solution (8) is unstable. Let us consider some particular cases.

1. A body supported at a point on a plane. In that case

$$\xi = 0, \qquad \xi = -a, \qquad a_1 = b_1 = b_2 = h_2 \equiv 0$$

$$a_2 = B / A_1, \qquad h_1 = s / A_1, \qquad A = A + M_a^2, \qquad B_1 = B$$

The known condition of stability for the regular precession of a rigid body with a fixed point

$$(Br_0 + s)^2 - 4MgA_1 \sin\alpha_0 > 0$$

follows from (12) by virtue of (4).

2. Linearly rolling body. Let $z'(\alpha^\circ) = 0$, i.e. for $\alpha = \alpha_0$ the center of gravity of the body is above the point of contact.

For $P_0 = 0$, Equation (9) is automatically satisfied.

The inequality (12) takes the form

$$(B_1 r_0 + s)(a_2 r_0 + h_1) + Mgz''(\alpha) > 0$$

In particular, for a wheel rolling linearly ($\alpha_0 = P_0 = 0$) the quantity $z(\alpha)$ is an even function of α ans $z'(0) = 0$). In that case $\zeta(0) = 0$, $\xi(0) = a$ is the radius of a wheel and $\rho = z(0) + z''(0)$ is the radius of curvature of the meridian

$$A_1 = A, \qquad B_1 = B + Ma^2$$

The condition of stability has the form [4]

$$(Br_0 + s)^2 (Br_0 + s + Ma^2 r_0) - MgaA(1 - \rho / a) > 0$$

3. Small regular precessions of a top. In that case

$$\alpha_0 = \tfrac{1}{2}\pi - \beta_0; \quad z'(\tfrac{1}{2}\pi) = 0, \qquad z(\tfrac{1}{2}\pi) + z''(\tfrac{1}{2}\pi) = \rho, \qquad z''(\tfrac{1}{2}\pi) = \rho l$$

Let $O(\beta_0^2)$ be a small quantity of the order of β_0^2. From (1) there follows that $P_0 \cot\beta_0$ is bounded. The condition of existence of arbitrary small precessions follows from (11) and has the form

$$D(\tfrac{1}{2}\pi, r_0) = \{[B + M\rho^2(1 - l)] r_0 + s\}^2 + 4Mg\rho l [A + M\rho^2(1 - l)^2] > 0$$

The inequality (12), by making use of (9), becomes

$$D(\tfrac{1}{2}\pi, r_0) + O(\beta_0) > 0$$

Thus along any $z(\alpha)$ it is possible to chose an $\epsilon > 0$ such that the precessions for $\tfrac{1}{2}\pi - \varepsilon < \alpha_0 < \tfrac{1}{2}\pi$ are stable.

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